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Bicubic Spline Interpolation in L-Shaped Domains

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Birkhoff and de Boor first posed the question of the existence of a convergent bicubic spline interpolation scheme for non-rectangular domains. In this paper that query is answered affirmatively for L-shaped domains. Specifically, it is shown that $\| s_f - f \| = O(h^r)$ where s_f is the bicubic spline interpolant associated with a smooth function f, h is the maximum mesh spacing, $r = 4$ for uniform partitions, and $r = 3$ for nonuniform partitions.

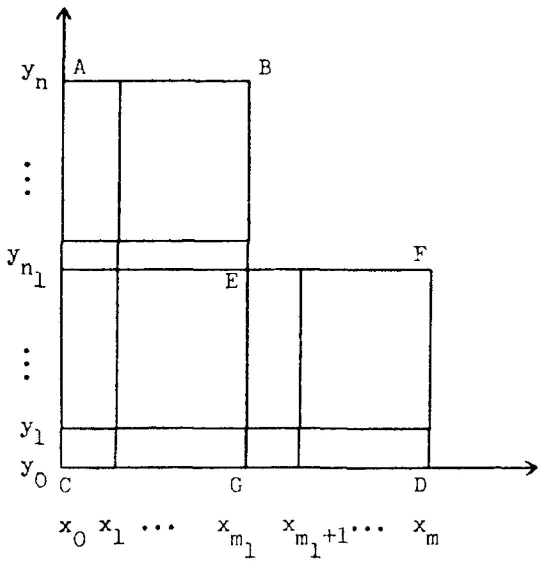
1. INTRODUCTION

The present paper deals with the problem of how to construct a convergent bicubic spline interpolation scheme for L-shaped domains. This problem was first posed by Birkhoff and de Boor [2, pp. 186–187]. One such method was described in detail by the present authors in an early draft of [5], and referred to by Professor Birkhoff in his survey article [1, Theorem 4]. The order of convergence was later improved (for nonuniform partitions) and was reported in [4, Theorem 9].

The purpose of this note it to indicate the proof of convergence for this bicubic spline interpolation scheme. Specifically, it is shown that $\| s_f - f \| = O(h^r)$ where s_f is the bicubic spline interpolant associated with a smooth function f, h is the maximum mesh spacing, $r = 4$ for uniform partitions and $r = 3$ for nonuniform partitions. In addition, results involving perturbation of (univariate) cubic splines are presented which are used in establishing this convergence and which are of interest in their own right.

2. A Convergent Interpolation Scheme

Let (L, π) denote the partitioned L-shaped domain in Fig. 1. The smooth Hermite space consisting of piecewise bicubic polynomials in $C^1[L]$ is denoted by $H^2(L, \pi)$.



$R_1 : [x_0, x_{m_1}] \times [y_0, y_n]$, $R_2 : [x_{m_1}, x_m] \times [y_0, y_{n_1}]$, $L = R_1 \cup R_2$

Figure 1

The spline subspace $S^2(L, \pi)$ is defined by

$$S^2(L, \pi) \equiv H^2(L, \pi) \cap C^2[L]. \tag{1}$$

It is well known [3] that given a function $f \in C^4[L]$, the smooth Hermite interpolant u_f, of f satisfies

$$\| u_f^{(k,l)} - f^{(k,l)} \|_\infty = O(h^{4-k-l}), \qquad 0 \leqslant k + l \leqslant 3^1. \tag{2}$$

where h is the maximum mesh spacing. The main result of this paper establishes that u_f in (2) essentially can be replaced by a bicubic spline of interpolation. This result is contained in the following theorem.

[1] $g^{(k,l)} = \partial g^{(k+l)}/\partial x^k \partial y^l$ and $\| g^{(k,l)} \|_\infty = \max_{i,j} \| g^{(k,l)} \|_{[x_i, x_{i+1}] \times [y_j, y_{j+1}]}$.

THEOREM 1. *Let $f \in C^5[L]$ where (L, π) is given in Fig. 1. Then,*

(i) *there exists a unique bicubic spline interpolate w_f of f satisfying*

(a) $w_f(x_i, y_j) = f(x_i, y_j)$ *at each mesh point* $(x_i, y_j) \in \pi$.

(b) $w_f^{(1,0)}(x_i, y_j) = f^{(1,0)}(x_i, y_j)$ *along* $\overline{AC}, \overline{BE}, \overline{FD}$ *and at the corners* A, B, C, D, F

(c) $w_f^{(0,1)}(x_i, y_j) = f^{(0,1)}(x_i, y_j)$ *along* $\overline{AB}, \overline{CD}$ *and at the corners* A, B, C, D

(d) $w_f^{(1,1)}(x_i, y_j) = f^{(1,1)}(x_i, y_j)$ *along* $\overline{EF}$ *and at the corners* A, B, C, D *and* F.

(ii) *If the maximum mesh ratios* $\beta = \max_i(x_{i+1} - x_i)/\min_i(x_{i+1} - x_i)$ *and* $\beta' = \max_j(y_{j+1} - y_j)/\min_j(y_{j+1} - y_j)$ *remain bounded as*

$$h = \max_{i,j}\{x_i - x_{i-1}, y_j - y_{j-1}\} \to 0,$$

then

$$\| w_f^{(k,l)} - f^{(k,l)} \|_\infty = O(h^{4-k-l}), \qquad 0 \leqslant k + l \leqslant 3 \text{ in } R_1, \tag{3}$$

and

$$\| w_f^{(k,l)} - f^{(k,l)} \|_\infty = O(h^{3-k-l}), \qquad 0 \leqslant k + l \leqslant 2 \text{ in } R_2. \tag{4}$$

(iii) *If* $\beta = \beta' = 1$ then (3) holds throughout L.

Proof. We consider here only the salient features of the proof; the interested reader is referred to [5] for the complete details.

It was shown in [4, Theorem 2] that any function s in $S^2(L, \pi) \equiv H^2(L, \pi) \cap C^2[L]$ is contained in $C^{(2,2)}[L]$. Thus, along each mesh line $x = x_i$ ($y = y_j$), the univariate functions $s^{(k,0)}(x_i, y)$, $0 \leqslant k \leqslant 2$, ($s^{(0,l)}(x, y_j)$, $0 \leqslant l \leqslant 2$) are cubic splines. The data given in (i)(a–d) uniquely define cubic splines along each mesh line (c.f. [4, 5]). Hence, the complete set of Hermite coordinates of a function w_f is determined. Since $w_f \in C^{(2,2)}[L]$, $w_f \in S^2(L, \pi)$ and the proof of (i) is complete.

Using the constructive sequence described in [5, p. 15] and Theorems 2 and 3 in the Appendix, one can derive error bounds for the Hermite coordinates of w_f, i.e. bounds for

$$w_f^{(k,l)}(x_i, y_j) - f^{(k,l)}(x_i, y_j) = w_f^{(k,l)}(x_i, y_j) - u_f^{(k,l)}(x_i, y_j), \quad 0 \leqslant k, l \leqslant 1.$$

Specifically, one can derive

$$|(w_f - u_f)^{(k,l)}(x_i, y_j)| = O(h^{s-(k+l)}), \qquad 0 \leqslant k, l \leqslant 1 \tag{5}$$

where $s = 4$ for $\beta = \beta' = 1$ and $s = 3$ otherwise. Thus, $(w_f - u_f)$ is a bicubic Hermite polynomial whose Hermite coordinates are converging to zero. Since the Hermite basis functions are sufficiently well-behaved as $h \to 0$ [7, p. 213] we have

$$\|(w_f - u_f)^{(k,l)}\|_\infty = O(h^{s-(k+l)}), \qquad 0 \leqslant k, l \leqslant 1. \tag{6}$$

The main result follows from (2), (6), and the triangle inequality

$$\|(w_f - f)^{(k,l)}\|_\infty \leqslant \|(w_f - u_f)^{(k,l)}\|_\infty + \|(u_f - f)^{(k,l)}\|_\infty . \tag{7}$$

APPENDIX

The following theorems establish bounds for the propagation of errors in the given data of two (univariate) cubic spline interpolation schemes. These results are used in the proof of Theorem 1 and are also of interest in the context of the stability of spline interpolation.

THEOREM 2. *Let $g \in C^k[I]$ where $I = [a, b]$ is partitioned by*

$$\pi : a = x_0 < x_1 < \cdots < x_n = b,$$

and let $s(x)$ be the cubic spline satisfying $s(x_i) = g(x_i) + \xi_i$, $0 \leqslant i \leqslant n$ and $s'(x_j) = g'(x_j) + \eta_j$, $j = 0, n$. If $|\xi_i| \leqslant K_1$ and $|\eta_j| \leqslant K_2$, then, for $1 \leqslant i \leqslant n-1$,

$$| s'(x_i) - f'(x_i)| \leqslant \begin{cases} 4/27 \| g^{(3)} \|_\infty h^2 + (6/\hat{h}) K_1 + K_2 , & k = 3 \\ 1/24 \| g^{(4)} \|_\infty h^3 + (6/\hat{h}) K_1 + K_2 , & k = 4 \\ 1/60 \| g^{(5)} \|_\infty h^4 + (6/\hat{h}) K_1 + K_2 , & k = 5, \\ & \textit{uniform partition} \end{cases} \tag{8}$$

where $\hat{h} = \min_{0 \leqslant j \leqslant n-1} | x_{j+1} - x_j |$.

Proof. For $K_1 = K_2 = 0$ (i.e., $s \equiv s_g$) and $k = 3$ or 4 it is shown in [7] that for $1 \leqslant i \leqslant n-1$

$$| s_g'(x_i) - g'(x_i)| \leqslant \begin{cases} 4/27 \| g^{(3)} \|_\infty h^2 & k = 3 \\ 1/24 \| g^{(4)} \|_\infty h^3 & k = 4. \end{cases} \tag{9}$$

In a similar manner for $k = 5$ and a uniform partition it can be shown that

$$| s_g'(x_i) - g'(x_i)| \leqslant 1/60 \| g^{(5)} \|_\infty h^4, \qquad 1 \leqslant j \leqslant n-1. \tag{10}$$

For K_1 and/or K_2 positive,

$$s(x) = s_g(x) + \epsilon(x) \tag{11}$$

where $\epsilon(x)$ is a cubic spline satisfying

$$\begin{aligned} \epsilon(x_i) &= \xi_i, \qquad 0 \leqslant i \leqslant n \\ \epsilon'(x_j) &= \eta_j, \qquad j = 0, n \end{aligned} \tag{12}$$

The Hermite coordinates of $\epsilon(x)$, satisfy [2]

$$[\Delta x_{i+1}\, \epsilon'(x_{i-1}) + 2(\Delta x_i + \Delta x_{i+1})\, \epsilon'(x_i) + \Delta x_i\, \epsilon'(x_{i+1})]$$

$$= 3\left[-\frac{\Delta x_{i+1}}{\Delta x_i}\, \epsilon(x_{i-1}) + \left(\frac{\Delta x_{i+1}}{\Delta x_i} - \frac{\Delta x_i}{\Delta x_{i+1}}\right) \epsilon(x_i) + \frac{\Delta x_i}{\Delta x_{i+1}}\, \epsilon(x_{i+1})\right] \tag{13}$$

for $1 \leqslant i \leqslant n-1$ where $\Delta x_j = x_j - x_{j-1}$.

Solving (13) for the unknown Hermite coordinates

$$\bar{E} = (\epsilon'(x_1), \epsilon'(x_2), ..., \epsilon'(x_{n-1}))^T$$

we obtain

$$M\bar{E} = \bar{\Phi} + \bar{\psi} \tag{14}$$

where $\bar{\Phi}$ and $\bar{\psi}$ are composed of appropriate terms involving ξ_i and η_j, respectively, and M is a tridiagonal matrix. Premultiplying (14) by the diagonal matrix $D = \operatorname{diag}\{d_{ii}\}$ where $d_{ii} = 1/2(\Delta x_i + \Delta x_{i+1})$ and inverting yields as in [7]

$$\| \bar{E} \| \leqslant \|(DM)^{-1}\| [\| D\Phi \| + \| D\psi \|]. \tag{15}$$

Since $\|(DM)^{-1}\| \leqslant 2$, $\| D\phi \| \leqslant (3/\hat{h})\, K_1$ and $\| D\psi \| \leqslant 1/2K_2$, [7, p. 213] we have

$$\| E \| \leqslant (6/\hat{h})\, K_1 + K_2 . \tag{16}$$

The proof of Theorem 2 follows from (9), (10), (16) and

$$| s'(x_i) - g'(x_i)| \leqslant | s_g'(x_i) - g'(x_i)| + | \epsilon'(x_i)|. \tag{17}$$

Q.E.D.

By considering s as an element of the smooth Hermite space one has the following.

COROLLARY 1. *If the mesh ratio* β *is bounded and* $K_1 = O(h)$ *and* $K_2 = O(1)$, *then s converges uniformly over I to g as* $h \to 0$.

Next, we have the following theorem.

THEOREM 3. *Let* $g \in C^5[I]$ *and let* $r(x)$ *be the unique cubic spline satisfying* $r(x_i) = g(x_i) + \epsilon_i$, $i = 0, 1$ *and* $r'(x_j) = g'(x_j) + \eta_j$, $0 \leqslant j \leqslant n$, *relative* *to the partition* π: $a = x_0 < x_1 < \cdots < x_n = b$. *If* $|\xi_i| \leqslant K_1$ *and* $|\eta_j| \leqslant K_2$, then for $2 \leqslant i \leqslant n$

$$|r(x_i) - g(x_i)| \leqslant K\hat{h}^3 + (K_1K_3 + K_2K_4)/\hat{h} \tag{18}$$

where

$$K = (\beta(b-a)/48)[\| g^{(4)} \| + (\beta(b-a)/3) \| g^{(5)} \|]$$

$$K_3 = [\hat{h}(1 + 2\beta^2) + \beta(\beta^2 - 1)(b - a)]$$

and $$K_4 = [\beta(b-a)\, h + (1/2)[(b-a)^2 + 2\hat{h}(b-a)](\beta^2 - 1)].$$

Further, if $\eta_i = 0$ *for* $i \geqslant 2$, *then*

$$K_4 = [(b-a)(\beta^2 - 1) + 2\beta h]\, \hat{h}.$$

If π *is uniform then*

$$K = [(b-a)/180] \| g^{(5)} \| \, h.$$

Proof. Consider first the case $K_1 = K_2 = 0$ ($r(x) \equiv r_g(x)$). For π uniform this result is essentially that of Loscalzo and Talbot [8, Theorem 5][2]. The analog for π nonuniform is established by considering the well-known relationship between the Hermite coordinates of $r(x)$, i.e. (13) with ϵ replaced by r.

From these equations we obtain the matrix equation

$$A\bar{Z} = \bar{b} \tag{19}$$

where A is lower triangular,

$$\bar{Z} = [r(x_3) - g(x_3), r(x_4) - g(x_4), \ldots, r(x_n) - g(x_n)]^T$$

and $\bar{b}$ involves higher order terms in $g^{(4)}(x_j)$ and $g^{(5)}(x_j)$. If (19) is premultiplied by a diagonal matrix $D = \text{diag}\{d_{ii}\}$ where $d_{ii} = 1/\Delta x_i\, \Delta x_{i+1}$ then $(DA)^{-1}$ is lower triangular and $[(DA)^{-1}]_{i+k,i} = \Delta x_{i+1}^2 - \Delta x_{i+2}^2 + \cdots + (-1)^k \Delta x_{i+k+1}^2$ for $0 \leqslant k \leqslant (m - 1 - i)$ and $1 \leqslant i \leqslant m - 1$. After some manipulation one can then establish [5, p. 8] that

$$\| Z \| \leqslant Kh^3 \tag{20}$$

For K_1 and/or K_2 positive $r = r_g + \epsilon$ and (19) becomes

$$A\bar{Z} = \bar{b} + \bar{\Phi} + \Psi \tag{21}$$

[2] Their result requires one more continuous derivative than stated in [8].

where $\bar{\Phi}$ and $\bar{\Psi}$ involve ξ_i and η_j respectively. Then

$$|\epsilon(x_i)| \leqslant \|(DA)^{-1}\| [\| D\bar{\Phi} \| + \| D\bar{\Psi} \|] \, (K_2 K_4 + K_1 K_3)/\bar{h}. \tag{22}$$

The main result then follows from (18), (22), and

$$| r(x_i) - g(x_i)| \leqslant | r_g(x_i) - g(x_i)| + | \epsilon(x_i)|. \qquad \text{Q.E.D.}$$

By considering r as an element of the smooth Hermite space one has the following.

COROLLARY 2. *If the mesh ratio β is bounded and $K_1 = O(h^2)$, $K_2 = O(h^2)$ then the cubic spline r converges uniformly over I to g. If π is uniform then K_1 and K_2 need only be $O(h)$.*

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